Wolfram Demonstration Projects to Simulate the Control of Vibrations on a String

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Overview

Introduction

Partial-Differential-Equation (PDE) Model

Approximations

Wolfram Demonstration Projects

Real World Applications

- Engineering applications:
  - Small airplane wings/winglets
  - Wind turbines
  - Piezoelectric acoustic devices i.e. sound ducts



Figures:

https://media.wired.com/photos/5955af5fad90646d424bb358/master/pass/GettyImages-498118341.jpg, https://media.wired.com/photos/5d019ecca59542160d9c6275/master/pass/science_wind-turbine_1130718980.jpg, https://thermaflex.net/wp-content/uploads/2016/03/thermaflex-m-ke-flexible-duct.jpg

Our Goal

Blind applications of well-known numerical techniques for the control of PDEs fail substantially.

Goals:

- To accurately discretize the PDEs for designing controllers. Filtering by adding a numerical viscosity term (damping) is a key!
- The system of PDEs may be complex and their discretizations can even be more complex.
- Even the simplest case can take computer programs a relatively long time to solve.
  - We try to make the simulations faster and more accurately using three different methods.

Vibrations on a string of length $L = 1$

Let $u(x, t)$ describe the shape of the centerline of the string at (x, t). Then, the equation of motion is described by:

$$\begin{cases} u_{tt} + \underbrace{k_1 u_t}_{\text{viscous (weak) damping}} - u_{xx} - \underbrace{h^2 u_{xxt}}_{\textit{filtering term}} = 0 \\ \\ \end{cases}$$

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Here $k_1 \in \mathbb{R}^+$, $k_3 \in \mathbb{R}^+$ are the viscous damping and boundary damping gains, respectively.

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The parameter h will be explained later.

Initial Conditions

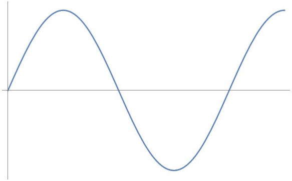
We have six different sets of initial conditions, i.e. initial shape and initial velocity of the string:

$$\begin{cases} u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x), \quad 0 \leq x \leq L \end{cases}$$

Initial Conditions

1. Sinusoidal-type

$$\begin{cases} u_0(x) = 10^{-3}\sin\left(\frac{(2k_4-1)\pi x}{2L}\right) \\ u_1(x) = 10^{-3}\sin\left(\frac{(2k_5-1)\pi x}{2L}\right), \\ 0 \le x \le L = 1 \end{cases}$$

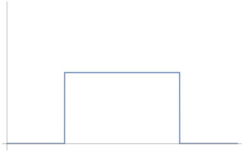


$k_4 \in \mathbb{R}^+$, $k_4 \in \mathbb{R}^+$ are given as the normal mode displacement coefficient and the normal mode velocity coefficient, respectively.

Initial Conditions

2. Box-type

$$\begin{cases} u_0(x) = \begin{cases} 10^{-3}, c_d - \frac{1}{4} < x < c_d + \frac{1}{4} \\ 0, x < c_d - \frac{1}{4} \text{ or } x > c_d + \frac{1}{4} \end{cases} \\ u_1(x) = \begin{cases} 10^{-3}, c_v - \frac{1}{4} < x < c_v + \frac{1}{4} \\ 0, x < c_v - \frac{1}{4} \text{ or } x > c_v + \frac{1}{4} \end{cases} \\ 0 \leq x \leq L = 1, \quad \frac{1}{4} < c_d < \frac{3}{4}, \quad \frac{1}{4} < c_v < \frac{3}{4} \end{cases}$$

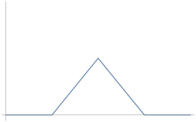


$c_d \in \mathbb{R}^+$ and $c_v \in \mathbb{R}^+$ are given as the centers of position and velocity of the box, respectively.

Initial Conditions

3. Pinch-type

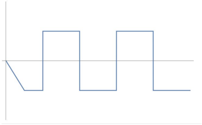
$$\begin{cases} u_0(x) = \begin{cases} \frac{1-4|c_d-x|}{10^3}, c_d - \frac{1}{4} < x < c_d + \frac{1}{4} \\ 0, x < c_d - \frac{1}{4} \text{ or } x > c_d + \frac{1}{4} \end{cases} \\ u_1(x) = \begin{cases} \frac{1-4|c_v-x|}{10^3}, c_v - \frac{1}{4} < x < c_v + \frac{1}{4} \\ 0, x < c_v - \frac{1}{4} \text{ or } x > c_v + \frac{1}{4} \end{cases} \\ 0 \leq x \leq L = 1, \quad \frac{1}{4} < c_d < \frac{3}{4}, \quad \frac{1}{4} < c_v < \frac{3}{4} \end{cases}$$



Initial Conditions

4. Square Wave Packets

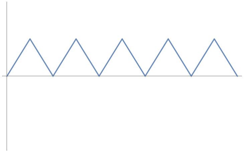
$$\begin{cases} u_0(x) = \dfrac{\left(2\left(\left\lfloor \frac{(2k_4-1)x}{2} \right\rfloor - 2\left\lfloor \frac{(2k_4-1)x}{4} \right\rfloor\right) - 1\right)}{10^3} \\ u_1(x) = \dfrac{\left(2\left(\left\lfloor \frac{(2k_5-1)x}{2} \right\rfloor - 2\left\lfloor \frac{(2k_5-1)x}{4} \right\rfloor\right) - 1\right)}{10^3} \\ 0 \leq x \leq L = 1 \end{cases}$$



Initial Conditions

5. Triangular Wave Packets

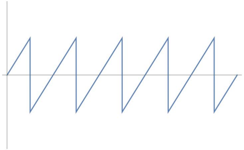
$$\begin{cases} u_0(x) = \frac{2\left|2\left(((2k_4-1)x)-\left\lfloor(2k_4-1)x+\frac{1}{2}\right\rfloor\right)\right|-1}{10^3} \\ u_1(x) = \frac{2\left|2\left(((2k_5-1)x)-\left\lfloor(2k_5-1)x+\frac{1}{2}\right\rfloor\right)\right|-1}{10^3} \\ 0 \leq x \leq L = 1 \end{cases}$$



Initial Conditions

6. Sawtooth Wave type

$$\begin{cases} u_0(x) = \dfrac{2\left((2k_4-1)\frac{x}{2}-\left\lfloor(2k_4-1)\frac{x}{2}\right\rfloor\right)-1}{10^3} \\ u_1(x) = \dfrac{2\left((2k_5-1)\frac{x}{2}-\left\lfloor(2k_5-1)\frac{x}{2}\right\rfloor\right)-1}{10^3} \\ 0 \leq x \leq L = 1 \end{cases}$$

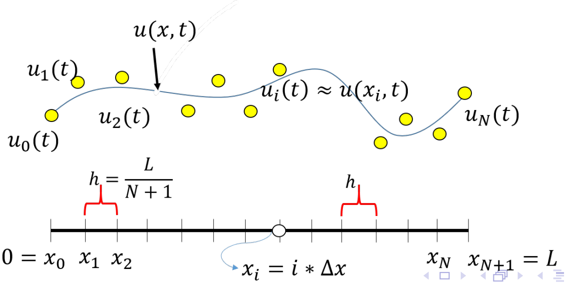


Finite Difference Method

Consider the finite-differences in space-discretization such that $u(x_j, t) \approx u_j(t)$, where $u(x_j, t)$ is the approximations of $u(x, t)$ at $x = x_j$. So, given $N \in \mathbb{N}$, we set $h = \frac{L}{N+1}$ to discretize the interval $[0, L]$ as follows:

$$x_0 = 0 < x_1 = h < \dots < x_N = Nh < x_{N+1} = L, \quad (1)$$

where $x_j = jh, j = 0, \dots, N+1$.



Discretized Model for the Wave Equation

Use the central difference formula at $x = x_j$
$u_{xx}(x_j, t) \approx \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2}$:

$$\begin{cases} u_{i,tt} + k_1 u_{i,t} - \frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = 0 \\ u(0,t) = u_{N+1} - u_N + k_3 \frac{1}{N+1} u_{N+1,t} = 0 \\ u_i(0) = 10^{-3} \sin\left(\frac{(2k_4-1)\pi i}{2N}\right) \\ u_{i,t}(0) = 10^{-3} \sin\left(\frac{(2k_5-1)\pi i}{2N}\right), \quad i = 1, 2, ..., N, \quad t \geq 0 \end{cases}$$

These discretizations are for the sinusoidal initial conditions, this can be replicated for the other types.

Implementing an Indirect Filtering

$$\begin{cases} u_{i,tt} + k_1 u_{i,t} - \frac{u_{i+1}-2u_i+u_{i-1}}{h^2} - k_2 \underbrace{h^2 \frac{u_{i+1,t} - 2u_{i,t} + u_{i-1,t}}{h^2}}_{filtering\ \ by\ \ a\ \ viscosity\ \ term} = 0 \\ u_0 = u_{N+1} - u_N + k_3 \frac{1}{N} u_{N+1,t} = 0 \\ u_i(0) = 10^{-3} sin(\frac{(2k_4-1)\pi i}{2N}) \\ u_{i,t}(0) = 10^{-3} sin(\frac{(2k_5-1)\pi i}{2N}), \quad i = 1, 2, ..., N, \quad t \geq 0 \end{cases}$$

Here, the control parameters are the same as before. We also add the control parameter k_2 which represents whether the system has a filtering term applied (k_2 is 0 or 1). Add this filtering term has been shown to make solving the system easier without changing the solutions.

Our Goal:

- Increasing the number of nodes N (or $h \to 0$) in the simulation for our discretizations, and try to make the simulations faster.

Finite Element Method (Linear Splines) - Zuazua-Tebou, 2007

Consider the following discretization of the finite element method:

$$\begin{cases} \underbrace{\frac{1}{6}u_{i+1,tt} + \frac{2}{3}u_{i,tt} + \frac{1}{6}u_{i-1,tt}} + k_1 u_{i,t} - \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - k_2 h^2 \frac{u_{i+1,t} - 2u_{i,t} + u_{i-1,t}}{h^2} = 0 \\ u(0,t) = u_{N+1} - u_N + k_3 \frac{1}{N} w_{N+1,t} = 0 \\ i = 1, 2, ..., N, \quad t \geq 0 \end{cases}$$

The main difference between the finite element method and the finite difference method is that the FEM works slower, but is overall more efficient than the FDM. Therefore, there is a tradeoff of efficiency with speed.

Order Reduction Finite Difference Scheme (unfiltered) - Liu, 2020

Consider the following discretization of the Guo method without filtering [1]:

$$\begin{cases} \underbrace{\frac{1}{4}u_{i+1,tt} + \frac{1}{2}u_{i,tt} + \frac{1}{4}u_{i-1,tt}} + k_1 u_{i,t} - \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0 \\ \underbrace{\frac{h}{4}u_{N+1,tt} + \frac{h}{4}u_{N,tt}} + \frac{u_{N+1} - u_N}{h} + \frac{k_3}{N}u_{N+1,t} = 0 \\ i = 1, 2, ..., N, \quad t \geq 0 \end{cases}$$

This method gets rid of the filtering term present in both of the equations, however it leads to a better efficiency of stabilization compared to the other two algorithms. The tradeoff is that it takes a lot longer to compute.

Wolfram Demonstrations

Past Code by D.J. Price (a former team member), Emma Moore (current team member).

- Wave Equation
- Beam Equation

Bibliography

J. Liu, B.-Z. Guo, A New Semidiscretized Order Reduction Finite Difference Scheme for Uniform Approximation of One-Dimensional Wave Equation, SIAM J. Control Optim., vol. 58, pp. 2256–2287, 2020.

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L. T. Tebou, E. Zuazua, Uniform boundary stabilization of the finite difference space discretization of the 1-d wave equation, Advances in Computational Mathematics, vol. 26, p. 337, 2006.

Thanks for your attention.
Any questions?

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