Robust Filtering of the Approximations for the Observability of Vibrations on a Three-Layer Beam

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Outline

Three-Layer Mead Marcus Beam Model

Let dots and primes denote the differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively.

$$\begin{cases} \ddot{z} + z'''' - B\phi' = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ -C\phi'' + P\phi = -Bz''', & (x,t) \in (0,L) \times \mathbb{R}^+ \\ z(0,t) = z(L,t) = z''(0,t) = z''(L,t) = 0, & t \in R^+ \\ \phi'(0,t) = \phi'(L,t) = 0, & t \in R^+ \\ z(x,0) = z_0(x), \quad \dot{z}(x,0) = z_1(x), & x \in (0,L) \end{cases} \tag{1}$$

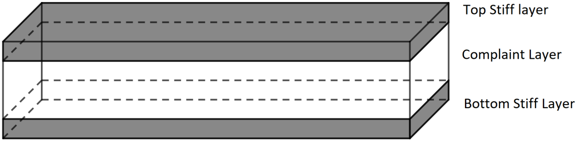

Figure: Three-layered beam

(1) can be rewritten as,

$$\begin{cases} \ddot{z} + (1 + B^2 J)z'''' = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ z(0,t) = z(L,t) = z''(0,t) = 0, \ z''(L,t) = 0, & t \in \mathbb{R}^+ \\ z(x,0) = z^0(x), \quad \dot{z}(x,0) = z^1(x), & x \in (0,L) \end{cases} \tag{2}$$

where $J = (-CD_x^2 + P)^{-1}$.

In comparison single-layer (Euler-Bernoulli) beam equation is,

$$\begin{cases} \ddot{z} + z'''' = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ z(0,t) = z(L,t) = z''(0,t) = 0, \ z''(L,t) = 0, & t \in \mathbb{R}^+ \\ z(x,0) = z^0(x), \quad \dot{z}(x,0) = z^1(x), & x \in (0,L) \end{cases} \tag{3}$$

The Exact Observability Inequality

The energy, $E(t)$, of (1) is defined as,

$$E(t) = \|(z, \dot{z})\|_E^2 = \frac{1}{2} \int_0^L |D_x^{-1} \dot{z}|^2 + |z'|^2 + (B^2 J z''') D_x^{-1} z dx \qquad (4)$$

(1) is a conservative system *i.e.* $\frac{\partial E(t)}{\partial t} = 0$, therefore $E(0) = E(t)\ \forall t \geq 0$.

The goal is to prove the exact observability by the following inequality:

$$\int_0^T |z'(L,t)|^2 dt \geq CE(0) \qquad (5)$$

Spectral Analysis

Two main methods used to prove (5) are:

- Multipliers Method (Komornik '97)
- Spectral Analysis (Non-harmonic Fourier series) (Komornik-Loreti '05)

The eigenvalues of (2) are $i\mu_k$ where,

$$\mu_k = \sqrt{1 + \frac{B^2}{C\lambda_k + P}\lambda_k}, \qquad \forall k > 0$$

$$\mu_k = -\mu_{-k} \qquad \forall k < 0$$

where $\lambda_k = \left(\frac{k\pi}{L}\right)^2$ are the eigenvalues of (3).

Therefore, the solutions to (2) may be expressed as

$$z(x,t) = \sum_{k \in \mathbb{Z}^*} a_k e^{i\mu_k t} \sin\left(\frac{k\pi x}{L}\right) \tag{6}$$

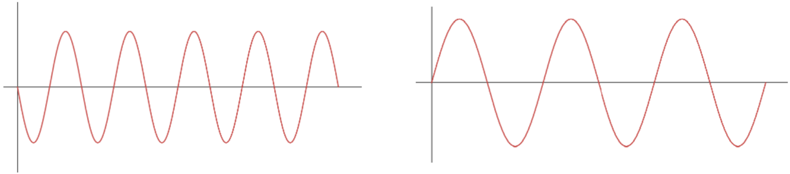

Figure: Two different Fourier modes of a solution at fixed times.

Idea: The uniform gap among eigenvalues of (3), $\inf_{m\neq n} |\lambda_m - \lambda_n| > 0$ can be used to show the uniform gap among the eigenvalues of (2), $\inf_{m\neq n} |\mu_m - \mu_n| > 0$

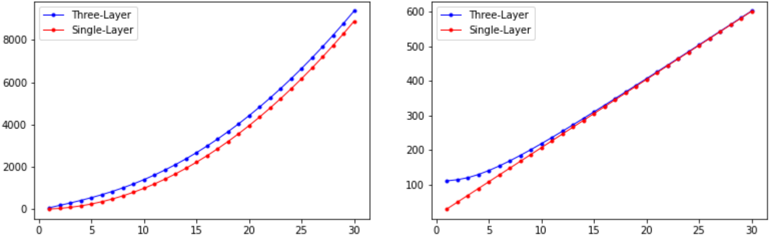

Figure: For $B = 10, C = 0.1, P = 1$ the eigenvalues (left) λ_k, and μ_k. Gap among consecutive eigenvalues (right).

Theorem

Let $g(x) : \mathbb{R}^+ \to \mathbb{R}^+$ *be a differentiable function such that* $g'(x) \geq 0$ *for all* $x \in \mathbb{R}^+$. *Let*

$$f(x) = \sqrt{1 + \frac{B^2}{Cg(x) + P}g(x)}.$$

Then for any $a \geq 0$,

$$f(x+a) - f(x) \geq g(x+a) - g(x) \geq 0$$

Corollary

Hence, $|\mu_m - \mu_n| \geq |\lambda_m - \lambda_n|$ *for all* $m, n \in \mathbb{Z}$

Theorem

For any $T > 0$ *there exist* $C = C(T) > 0$ *such that*

$$\int_0^T |z'(L,t)|^2 dt \geq CE(0) \tag{7}$$

Let $N \in \mathbb{N}$ be given, and define mesh parameter $h := \frac{L}{N+1}$. Consider the following discretization of the interval $[0, L]$:

$$0 = x_0 < x_1 < ... < x_j = jh < ... < x_{N-1} < x_N < x_{N+1} = L$$

Let $z_j = z_j(t) \approx z(x_j, t)$, and $\vec{z} = [z_1, z_2, ..., z_N]^T$.

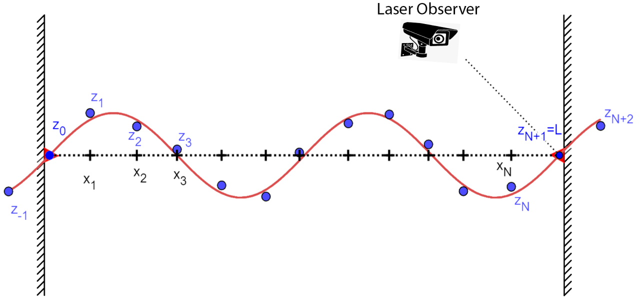

Figure: Space discretization of the $z(x, t)$

A Special Matrix A_h

Consider the Finite-Difference approximation of Laplace equation,

$$z_j'' \approx \frac{z_{j+1} - 2z_j + z_{j-1}}{h^2}$$

Therefore,

$$D_x^2 \vec{z} \approx -A_h \vec{z}$$

where

$$A_h = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{bmatrix} \tag{8}$$

The space discretization of (2) is,

$$\begin{cases} \ddot{\vec{z}} + (I + B^2(CA_h + PI)^{-1})A_h^2\vec{z} = 0, & t \in \mathbb{R}^+ \\ z_0 = z_{N+1} = 0, & t \in \mathbb{R}^+ \\ z_{-1} = -z_1, \;\; z_{N+2} = -z_N, & t \in \mathbb{R}^+ \end{cases} \tag{9}$$

The energy of (9) defined as,

$$\begin{aligned} E_h(t) = \frac{h}{2}\sum_{j=1}^{N} & \left| \frac{(A_h^{-1}\dot{\vec{z}})_{j+1} - (A_h^{-1}\dot{\vec{z}})_j}{h} \right|^2 + \left| \frac{z_{j+1} - z_j}{h} \right|^2 \\ & + B^2 \left(\frac{(J_hA_h\vec{z})_{j+1} - (J_hA_h\vec{z})_j}{h} \right) \left(\frac{(A_h^{-1}\vec{z})_{j+1} - (A_h^{-1}\vec{z})_j}{h} \right) \end{aligned} \tag{10}$$

where $J_h = (CA_h + PI)^{-1}$.

The eigenvalues of (9) are $i\mu_k(h)$ where,

$$\mu_k(h) = \sqrt{1 + \frac{B^2}{C\lambda_k(h) + P}}\lambda_k(h), \qquad k = 1, 2, ..., N$$
$$\mu_k(h) = -\mu_{-k}(h), \qquad k = -1, -2, ..., -N$$

where $\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right)$ are the eigenvalues of A_h.

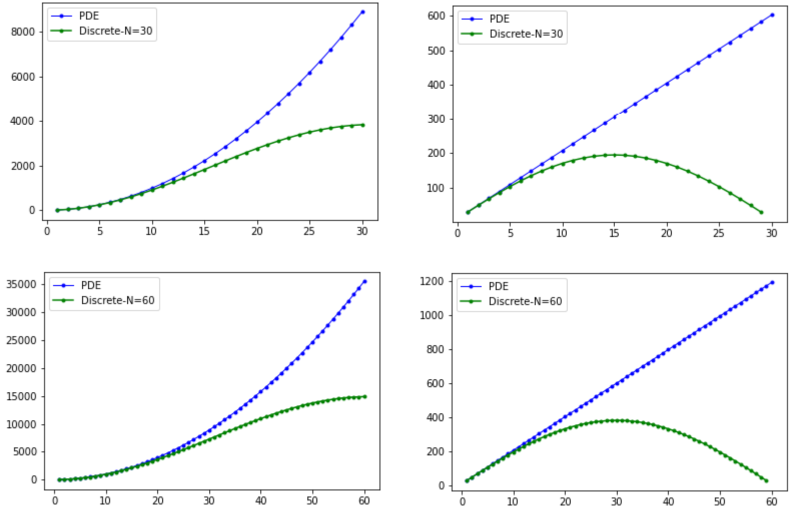

Figure: For $B = C = P = 1$, the first $N = 30$ (top) and $N = 60$ (bottom) eigenvalues of (2) μ_k, and (9) $\mu_k(h)$ and gap among consecutive eigenvalues (right)

Lack of Uniform Observability

By using the identity (Infante-Zuazua '99)

$$\frac{h\sum_{j=0}^{N}\left|\frac{\phi_{k,j+1}-\phi_{k,j}}{h}\right|^2}{\left|\frac{\phi_{k,N}}{h}\right|^2}=\frac{2L}{4-\lambda_k(h)h^2} \tag{11}$$

where $\phi_{k,j}=\sin\left(\frac{jk\pi h}{L}\right)$.

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where $\phi_{k,j} = \sin\left(\frac{jk\pi h}{L}\right)$.Observe that $\lambda_N h^2 \to 4$ as $h \to 0$.

Theorem

For any $T > 0$,

$$\lim_{h\to 0} \sup_{z sol.\ of\ (9)} \frac{E_h(0)}{\int_0^T \left|\frac{z_N}{h}\right|^2 dt} \to \infty \tag{12}$$

Direct Filtering

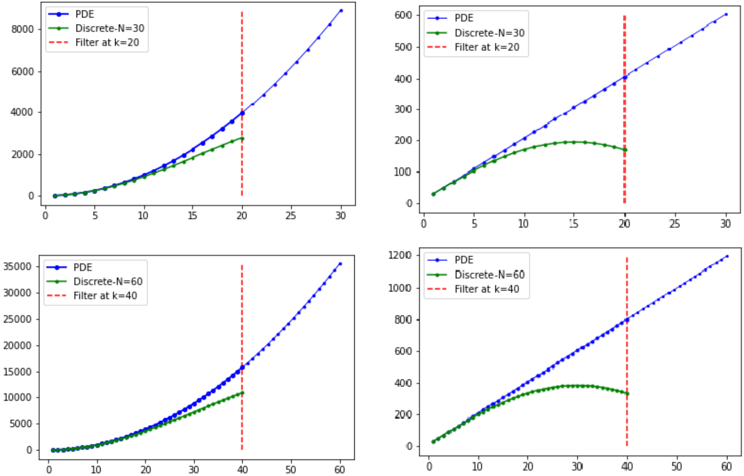


Figure: For $B = C = P = 1$, the first $N = 30$ (top) and $N = 60$ (bottom) eigenvalues of (2) μ_k, and (9) $\mu_k(h)$ and gap after direct filtering.

Given any $\gamma \in (0,4)$ define the following solution space

$$\boldsymbol{C}_h(\gamma) = \left\{ \vec{z}_h \text{ sol. of (9)} \mid \vec{z}_h = \sum_{\lambda_k(h)h^2 \leq \gamma} (a_k e^{i\mu_k(h)t})\vec{\phi}_k(h) \right\} \tag{13}$$

Theorem

Let $\gamma \in (0,4)$, for any $T > 0$ there exist $C = C(T,\gamma) > 0$ such that

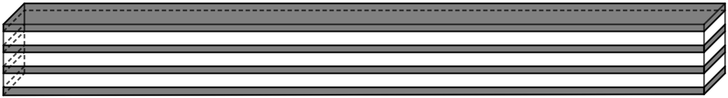
$$\int_0^T \left| \frac{z_N}{h} \right|^2 dt \geq CE(0), \qquad \forall \vec{z}_h \in \boldsymbol{C}_h(\gamma) \tag{14}$$

Future Work

- Explicit representation of eigenvalues is not possible for some other boundary conditions. Therefore the Multipliers method is necessary to be utilized.
- The Mead-Marcus beam model has been generalized for arbitrary number of layers. For $n = 2m + 1$ layers,

$$\begin{cases} \ddot{z} + z'''' - B^T \vec{\phi}' = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ -C\vec{\phi}'' + P\vec{\phi} = -Bz''', & (x,t) \in (0,L) \times \mathbb{R}^+ \end{cases}$$

where B is a $m \times 1$ column vector with positive entries, P and C are $m \times m$ invertible, symmetric, positive definite matrices.



- Finding optimal filtering parameter γ is an important requirement for the real life applications.

- Recent studies show that the exact observability can be retained without any filtering by the use of "Order reduced Finite-Difference scheme". Similar approximation can be applied to the multi-layer case. (Guo-Liu '20)

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